Lecture Notes Supplement on Growth Theory
Economics of the Environment and Natural Resources/Economics of Sustainability
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There are various references to growth theory in the papers that we've been reading on the economics of climate change. I've gone lightly since these are not essential to the argument, but for those who are more interested these notes might help point the way.

First recall the issue of discounting, which we touched upon earlier. Individuals have a preference for consumption now over consumption later: we're impatient. This means that, given a choice between $100 paid now and more money paid a year later, people will accept less money now. A patient person might give up no more than $101 next year; an impatient person might give up $110 or even more. (By "impatient" we lump together a mass of characteristics from what psychologists call "present oriented" to executive functions and impulse control to external constraints.)

We see vast proof of people's willingness to make this trade by, for example, the fact that people routinely take on debts: holding a credit card balance at an interest rate of 18% means that people take $100 today and pay $118 or more next year.

The broader question, however, is whether policymakers ought to discount in this way. Is it ethical for a society to take on expensive debts? (Again, many governments do. However this is irrelevant to deontology.) This question is large and multi-faceted; a paragraph cannot do justice to either side of the argument. To make the problem most pointed: some government spending can save lives so a discount rate, applied to government spending choices, means that government is willing to save fewer than 100 lives today, in return for sacrificing 100 lives in the future. These sorts of questions have dogged philosophers for ages and we've mostly abandoned any hopes of coming up with a solution that could be broadly agreed upon. (Ethical questions are commonly put in railroad terms: you control a switch that can change the track upon which a runaway locomotive will roll; would you switch from killing 2 people to killing one person? What if the act of controlling the switch involved murdering someone? This is how philosophers while away the hours.) But the lack of clear moral guidance about the single right choice does not allow us to postpone these decisions.

Government policy chose to build transportation infrastructure in NYC such as airports and highways, which increase current well-being, at the expense of poverty-reduction or poverty-alleviation in the past. Was that right? Is it better, if the government has $1bn dollars to spend, to vaccinate children or build bridges or abate CO_2 emissions?

In all of this, we note that governments must make choices to spend more money now even if it means spending less money later. We attempt to describe this trade-off with discount rates.

For an individual, borrowing with a credit card means that \$100 today is preferred to \$118 next year. If the individual borrows that money next year, then they would trade \$118 next year for (1 + .18)118 in two years – which is (1 + .18)(1 + .18)100. The general formula, for an interest rate r and time T, is that a person chooses between 100 and $(1 + r)^T$ 100. There is some rate (a discount factor, usually noted β) that makes 100 = $\beta^T (1 + r)^T$ 100, so the discount factor β=1/(1+r).

A higher interest rate means that future outcomes receive less weight; you can think of it as a "hurdle rate" for public projects. If the future is discounted at 4%, fewer projects will clear the hurdle than if the rate is 2% or 1%.

Growth Theory

This notion can be put into more formal economic theory as well. We want to look at how people decide what income to consume and what to save – where the savings are invested, which increases the amount of capital and so increases the amount of income that can be produced in later periods. By "capital" we can mean physical capital like machines, or human capital like education or knowledge, or environmental capital like clean air and water. But the same basic decision would have to be made by even a simple agricultural community, trying to decide how much wheat to eat and how much to save to plant for the next season.

Assume that people get instantaneous utility in each time period from that period's consumption, represented by a CES function $\frac{c^{1-\sigma}-1}{1-\theta}$.

People do not only maximize their utility at one time; they look into the future and maximize their lifetime utility, subject to discounting as discussed above. But people take decisions about events even beyond their lifetime, either because they care about their children (if they have them) or they care about posterity. This can be modeled most simply by abandoning actuarial standards and assuming that people look all the way into the infinite future. (While this assumption sounds a bit crazy, discounting means that events more than a century away have almost no impact anyway.)

For mathematical simplicity we'll work in continuous time and assume that discounting is done continuously so with a discount rate of ρ, future values are discounted by an amount exp(-ρt).

So people don't just maximize their utility at any moment but their utility over the whole future; write this as $U(0)=\int_0^\infty e^{-\rho t}\frac{c^{1-\sigma}-1}{1-\sigma}dt$ subject to their budget constraint.

The budget constraint is a formalism of the problem stated before: income (y) that is not consumed is invested in creating new capital, so I = y – c. Capital (k) depreciates at a rate δ. So the change in capital stock is given as Δk = I – δk = y – c – δk. All of these are per-capita values.

The greater the stock of capital, the greater the output that can be made. Assume y=Ak, where A is some parameter. Rewrite the capital accumulation constraint as Δk = y – c – δk = Ak – c – δk. The marginal product of capital is important since firms invest as long as the marginal product of capital is greater than the interest rate.

The problem then is to max $U(0)=\int_0^\infty e^{-\rho t}\frac{c^{1-\sigma}-1}{1-\sigma}dt$ subject to Δk = Ak – c – δk. This is a problem in optimal control, of dynamic constrained optimization, which can be solved in a manner similar to the Lagrangian method of single-period optimization, known as the Hamiltonian method. (Details below.) It's a bit of calculus with long algebra.

The bottom line is that the growth rate of consumption, %Δc, is a function of the discount rate, elasticity of substitution, and the net productivity of capital; the formula is $\%\Delta c=\frac{1}{\sigma}(A-\delta-\rho)$.

A somewhat more interesting model can take account of public goods and the inefficiency of private provisioning of public goods. This model was developed in Barro (1990) in J. of Political Economy, also see Sala-i-Martin (1992). That is in the next section.

Hamiltonian (very simplified!)

Given a problem that has a control variable and a state variable, such as max $\int_0^\infty u(x)dt$ (where x is the control) subject to constraints that tell how the state variable changes over time, $g(x,k)=0$ (so k is the state variable), set up a "Hamiltonian" as $H\equiv u(x)+\lambda\left[g(x,k)\right]$. The maximum (given a bunch of technical transversality considerations, ignored for now) is found by setting $\frac{dH}{dx}=0$ and $\frac{dH}{dk}=-\dot{\lambda}$ where we use Newton's notation that a dot over a variable is its change over time, $\dot{\lambda}=\frac{d\lambda}{dt}$.

For this problem, the constraint for the households is that Δk = Ak – c – δk; write this as $\dot{k}=Ak-c-\delta k$. The tension is that people want to eat more now (more c) but that means less c in the future because k will be lower now. The optimal balance is found with the Hamiltonian.

Set up $H=e^{-\rho t}\frac{c^{1-\sigma}-1}{1-\sigma}+\lambda\left[Ak-c-\delta k\right]$ so $\frac{dH}{dc}=0=e^{-\rho t}c^{-\theta}-\lambda$ and $\frac{dH}{d\lambda}=-\dot{\lambda}=\lambda\left[A-\delta\right]$.

For the first equation, we have a "trick" that notes that the derivative of the natural log function is the percent growth, $d\left(\ln(\lambda)\right)=\frac{\dot{\lambda}}{\lambda}$, so from $\lambda=e^{-\rho t}c^{-\sigma}$, $\ln\lambda=-\rho t-\sigma\ln c$ so

$\frac{\dot{\lambda}}{\lambda} = -\rho - \sigma\frac{\dot{c}}{c}$. The second equation can be re-written as $-\frac{\dot{\lambda}}{\lambda} = [A - \delta]$. So set the two equations equal (which gets rid of those λ values) and solve to get the steady-state growth rate of consumption, that $\frac{\dot{c}}{c} = \frac{1}{\sigma}(A - \delta - \rho)$.

Growth Theory with a Public Good

Assume the following:

$y = k\phi\left(\frac{g}{k}\right)$, so per-capita income (y) is a function (with constant returns to scale) of per-capita capital stock (k) as well as some function, $\phi(\bullet)$, of the size of the per-capita public good (g) relative to capital. It is often convenient to assume a Cobb-Douglas production function that $y = Ak\left(\frac{g}{k}\right)^{\alpha}$. We leave out any decisions about labor supply – each firm has one worker.

The public good, g, can be considered as clean air (where "clean" includes a lack of too much CO2 that would result in warming) or g could be a healthy climate. Or g could be education or R&D or national defense or many other public goods. The key point is that it is not provided by an individual but only by a society acting together.

The government pays for the public goods by taxing output at a rate, τ ,to balance its budget so $g = \tau y$.

Firms invest in capital as long as the (after-tax) marginal productivity of capital is greater than its net cost, $(r + \delta)$, where δ is the depreciation rate and r is the real interest rate. By usual arguments, then, the marginal product of capital in equilibrium will equal $(r + \delta)$; take the derivative of y with respect to k to find the (after tax) marginal product of capital = $(1-\tau)\phi\left(\frac{g}{k}\right)\left[1 - \left(\frac{g}{y}\right)\phi'\right]$.

The consumers get instantaneous utility from consumption as $\frac{c^{1-\sigma} - 1}{1-\sigma}$ and so maximize their welfare, $U(0) = \int_0^{\infty} e^{-\rho t}\frac{c^{1-\sigma} - 1}{1-\sigma}dt$ subject to their budget constraint.

Solve the optimal control problem using the Hamiltonian method of dynamic constrained optimization. Details below.

The essential result is that there is an optimal level of governmental provision of public goods, that balances the disincentive effects of taxation (growth is lower because firms pay taxes on their output) with the positive effects of public goods.

So assessing if government investment crowds out private investment (in the sense of doing net harm to the economy) comes back to deciding if government investment returns at a higher rate than the private sector investment.

You could make it more complicated by introducing private provision of public goods or by introducing population explicitly, e.g. as $Y = AK^{\alpha} L^{1-\alpha} G^{1-\alpha}$.

Hamiltonian

For this problem, the constraint for the households is that capital changes by investment less depreciation, so $\dot{k} = (1-\tau) k \phi\left(\frac{g}{k}\right) - c - \delta k$. Set up $H = e^{-\rho t} \frac{c^{1-\sigma} - 1}{1-\sigma} + \lambda\left[(1-\tau) y - c - \delta k\right]$ = $H = e^{-\rho t} \frac{c^{1-\sigma} - 1}{1-\sigma} + \lambda\left[(1-\tau) k \phi\left(\frac{g}{k}\right) - c - \delta k\right]$. Set $\frac{dH}{dc} = 0 = e^{-\rho t} c^{-\sigma} - \lambda$ and $\frac{dH}{d\lambda} = -\dot{\lambda} = \lambda\left[(1-\tau)\phi\left(\frac{g}{k}\right)\left(1 - \frac{g}{k}\phi'\right) - \delta\right]$.

For the first equation, again use the "trick" that $d\left(\ln(\lambda)\right) = \frac{\dot{\lambda}}{\lambda}$, so $\lambda = e^{-\rho t} c^{-\sigma}$, $\ln\lambda = -\rho t - \sigma \ln c$ thus $\frac{\dot{\lambda}}{\lambda} = -\rho - \sigma \frac{\dot{c}}{c}$. The second equation can be re-written as $-\frac{\dot{\lambda}}{\lambda} = \left[(1-\tau)\phi\left(\frac{g}{k}\right)\left(1 - \frac{g}{k}\phi'\right) - \delta\right]$.

Set the two equations equal (which gets rid of those λ values) and solve to get the steady-state growth rate, that $\frac{\dot{c}}{c} = \frac{1}{\sigma}\left(MPK - \delta - \rho\right)$ where $MPK = (1-\tau)\phi\left(\frac{g}{k}\right)\left(1 - \frac{g}{k}\phi'\right)$.

Want more? Here's a good overview, http://ocw.mit.edu/courses/economics/14-451-macroeconomic-theory-i-spring-2007/lecture-notes/notes_ch_3.pdf